

Universal Teichmüller Space

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Abstract

We present an outline of the theory of universal Teichmüller space, viewed as part of the theory of QS , the space of quasimetric homeomorphisms of a circle. Although elements of QS act in one dimension, most results about QS depend on a two-dimensional proof. QS has a manifold structure modelled on a Banach space, and after factorization by $PSL(2, \mathbb{R})$ it becomes a complex manifold. In applications, QS is seen to contain many deformation spaces for dynamical systems acting in one, two and three dimensions; it also contains deformation spaces of every hyperbolic Riemann surface, and in this naive sense it is universal. The deformation spaces are complex submanifolds and often have certain universal properties themselves, but those properties are not the object of this paper. Instead we focus on the analytic foundations of the theory necessary for applications to dynamical systems and rigidity.

We divide the paper into two parts. The first part concerns the real theory of QS and results that can be stated purely in real terms; the basic properties are given mostly without proof, except in certain cases when an easy real-variable proof is available. The second part of the paper brings in the complex analysis and promotes the view that properties of quasimetric maps are most easily understood by consideration of their possible two-dimensional quasiconformal extensions.

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Introduction

The origins of this theory lie in the study of deformations of complex structure in spaces of real dimension 2 and the moduli problem for Riemann surfaces. It seems appropriate, therefore, to begin with a brief sketch of how the notion of a Teichmüller space first arose, within this problem of variation of complex structure on a topologically fixed compact Riemann surface. For brevity we shall restrict attention to *hyperbolic* Riemann surfaces, which have as universal covering space the unit disc; the terminology refers to the fact that, via projection from Poincaré's Riemannian metric on the disc, all the surfaces are endowed with a structure of hyperbolic geometry.

The definition of Teichmüller space stands out clearly as a key stage in the struggle to justify, and to make precise, the famous assertion of Riemann [*Theorie der Abel'schen Functionen*, Crelle J., t. 54, (1857)] that the number of (complex) parameters (or 'moduli') needed to describe all surfaces of genus $g \geq 2$ up to conformal equivalence is $3g - 3$. After preliminary work over many years by a substantial number of eminent mathematicians, including F. Schottky, A. Hurwitz, F. Klein, R. Fricke and H. Poincaré, the crucial new idea was introduced by O. Teichmüller around 1938, [71], [72], following earlier work of H. Grötzsch. One specifies a *topological marking* of the base surface and then considers all homeomorphisms to a target Riemann surface which have the property that they distort the conformal structure near each point by at most a bounded amount, using a precise analytic measure of the distortion to be defined below. Grötzsch (see [37],[38]) had used this approach to resolve similar problems in estimating distortion for smooth mappings between plane domains; the term *quasiconformal* was coined by L.V. Ahlfors around 1930 for the class of homeomorphisms to be employed. The method was strengthened, generalised and applied to the case of closed Riemann surfaces with striking effect by Teichmüller, as we indicate below.

A fundamental relationship exists between the quasiconformal homeomorphisms of the hyperbolic disc and the induced boundary maps of the circle, and this lies at the heart of the viewpoint on Teichmüller theory to be presented here: for a general Riemann surface, one must consider not only deformations

of complex structure in the interior but also the ways in which the conformal structure may change relative to the boundary. It turns out that both aspects are best studied on the universal covering surface, the unit disc $\Delta = \{|z| < 1\}$: quasiconformal mappings of the disc extend to homeomorphisms of the closed disc and many (but not all) of the properties of a quasiconformal homeomorphism can be expressed solely in terms of the boundary homeomorphism of the circle induced by it.

Let QS be the space of sense-preserving, quasisymmetric self-maps of the unit circle; such maps are precisely those occurring as the boundary value of some quasiconformal self map of the disc Δ . A map $H : \Delta \rightarrow \Delta$ is called *quasiconformal* (sometimes abbreviated to q-c) if $K(H) < \infty$, where $K(H)$ is the essential supremum, for $z \in \Delta$, of the local dilatations $K_z(H)$, and the *local dilatation* $K_z(H)$ at z is defined as

$$\limsup_{\epsilon \rightarrow 0} \frac{\max_{\theta} \{|H(z + \epsilon e^{i\theta}) - H(z)|\}}{\min_{\theta} \{|H(z + \epsilon e^{i\theta}) - H(z)|\}},$$

which may be interpreted as the upper bound of local distortion as measured on circles centered at z ; compare with the definition (18) in §2. The set of all possible quasiconformal extensions $H : \Delta \rightarrow \Delta$ of a given quasisymmetric map h may be regarded as the *mapping class* of h in the disc, and a mapping H_0 is called *extremal* for its class if $K(H_0) \leq K(H)$ for every extension H of h . This notion of extremality for a mapping (within a homotopy class of quasiconformal maps between two plane regions) was also introduced by Grötzsch (op.cit.), but it was Teichmüller who recognized the significance of extremal maps in the study of deformations of complex structures. He applied them decisively in [71], [72], to establish a measure of distance between two marked surfaces: here the upper bound for the local distortion of the mapping over the base surface is to be minimised.

A base (hyperbolic) Riemann surface is given as the quotient space $X_0 = \Delta/\Gamma$ of the unit disc under the action of a Fuchsian group Γ , which is by definition a discrete group of Möbius transformations which are conformal automorphisms of the disc; if the group is torsion-free, then topologically, $\Gamma \cong \pi_1(X_0)$ represents the group of deck transformations of the universal covering projection from Δ to X_0 . Suppose now that we are given a quasisymmetric map h of the circle with the property that the conjugate group $\Gamma_1 = h \circ \Gamma \circ h^{-1}$ is also Fuchsian: the two orbit spaces Δ/Γ and Δ/Γ_1 can be viewed as the same topological surface but with different complex structures. The *mapping class* of h for the group Γ is the subset of its mapping class (the set of *all* q-c self-mappings of the disc extending the map h) consisting of those q-c extensions H of h with the property that every element $H \circ \gamma \circ H^{-1}$ of Γ_1 acts as a Möbius transformation of the disc Δ to itself. For a Fuchsian group Γ that covers a compact Riemann surface, Teichmüller's theorem establishes a profound link between an extremal representative H for a given class and a holomorphic quadratic differential for Γ - a complete proof is given in [6]. As a consequence, one may infer that the space of marked deformations of the compact surface Δ/Γ is a complete metric space

homeomorphic to an $n = (6g - 6)$ -dimensional real cell. The metric is called Teichmüller's metric and the distance between the base surface $X_0 = \Delta/\Gamma$ and the marked surface $X_1 = \Delta/\Gamma_1$, with $\Gamma_1 = H_0 \circ \Gamma \circ H_0^{-1}$, is $\log K(H_0)$, where H_0 is extremal in its class. This type of extremal mapping is a feature of continuing interest, partly because of the connection with Thurston's theory of measured laminations on hyperbolic surfaces, [10],[18], [41], [30], [40],[68].

The final ingredient, which makes it possible to construct these holomorphic parameter spaces for all types of Riemann surface, is the relationship between the quasiconformal property and the solutions of a certain partial differential equation. By a fundamental observation of Lipman Bers (see [3]), if H is a quasiconformal self-map of the disc, it satisfies the Beltrami equation

$$H_{\bar{z}}(z) = \mu(z)H_z. \quad (1)$$

where μ , with $\|\mu\|_\infty < 1$, is a measurable complex-valued function on the disc, which represents the complex dilatation at each point of Δ ; μ is often called the *Beltrami coefficient* of H . Conversely, by virtue of solvability properties of this equation, μ determines H uniquely up to postcomposition by a Möbius transformation. By using the analytic dependence of $H = H_\mu$ on its Beltrami coefficient μ , and deploying a construction known as the Bers embedding, each $T(\Gamma)$ is embedded as a closed subspace of the complex Banach space B of univalent functions on Δ which have quasiconformal extensions to the sphere – more details are given in section 2.5. It then follows that $T(\Gamma)$ has a natural structure of complex manifold for *any* Fuchsian group Γ . Furthermore, each inclusion $\Gamma' \subset \Gamma$ of Fuchsian groups induces a contravariant inclusion of these Teichmüller spaces $T(\Gamma) \subset T(\Gamma')$, which implies that the Banach space $T(1) = B$, which corresponds to the trivial Fuchsian group $\Gamma' = 1 = \langle \text{Id} \rangle$ is *universal* in the sense that it contains the Teichmüller spaces of every hyperbolic Riemann surface Δ/Γ .

In the period after World War II, the verification of Teichmüller's ideas and the subsequent rigorous development of the foundational complex analytic deformation theory outlined above by L. V. Ahlfors, L. Bers, H.E. Rauch and their students occupied more than 20 years. The circumstances of Teichmüller's life and particularly his political activities caused much controversy and, coupled with the relative inaccessibility of his publications, this perhaps contributed to some early reluctance to pursue a theory based on his claims; for commentary on mathematical life in Germany under the Third Reich, the reader might consult [74] and [75]. Detailed expositions of this foundational work on moduli are given in [30], [17], [42], [57] and [53].

In a landmark study of the local complex analytic geometry of Teichmüller space, H.L. Royden [64] showed that when $T(\Gamma)$ is finite dimensional, the complex structure of the space determines its Teichmüller metric. In fact, he proved that Teichmüller's metric coincides with the Kobayashi metric [45], which is defined purely in terms of the set of all holomorphic maps from the unit disc into $T(\Gamma)$. Royden also showed that every biholomorphic automorphism of $T(\Gamma)$ is induced geometrically by an element of the mapping class group, a result

which extends to many infinite dimensional Teichmüller spaces; we examine this important rigidity theorem more carefully in sections 1.8 and 2.8.

The case of compact Riemann surfaces and their deformation spaces calls for techniques involving aspects of surface topology and geometry which will not be considered in this article. Instead, we present a formulation which focusses on the real analytic foundations of the theory, important for applications to real and complex dynamical systems and matters which relate to rigidity. It was observed by S.P. Kerckhoff (see for instance [76]) and later, independently, by S. Nag and A. Verjovsky [58] that the almost complex structure on each $T(\Gamma)$ corresponding to its complex structure is given by the Hilbert transform acting on the relevant space of vector fields defined on the unit circle. This fact indicates that deep results concerning the complex structure of Teichmüller space can be viewed purely as theorems of real analysis. With this principle in mind, we divide the exposition into two parts. The first part concentrates on the real theory of QS and we present the theorems in real terms as far as possible; the basic properties are stated mostly without proof, except in certain cases where an easy real-variable proof is available. The second part of the paper follows closely the outline of the first but brings in the complex analysis: in our view, despite their very real nature, properties of quasiconformal maps are most easily understood by consideration of their possible two-dimensional quasiconformal extensions.

1 Real Analysis

1.1 Quasisymmetry

A quasisymmetric map h of an interval I to an interval J is an increasing homeomorphism h for which there exists a constant M such that

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M \quad (2)$$

for every x and $t > 0$ with $x-t, x$ and $x+t$ in I . It is not hard to prove the quasisymmetric maps form a pseudo-group. That is, if h is quasisymmetric from I to J with constant M , then h^{-1} from J to I is quasisymmetric with constant M_1 depending only on M . Moreover, if g is quasisymmetric from I_1 to I_2 with constant M_g and h is quasisymmetric from I_2 to I_3 with constant M_h , then $h \circ g$ is quasisymmetric from I_1 to I_3 with constant $M_{h \circ g}$ depending only on M_g and M_h . Also, h is Hölder continuous with Hölder exponent α depending only on M . For purposes of illustration, we prove this fact here.

Lemma 1 *A quasisymmetric map of an interval I to an interval J satisfying condition (2) is Hölder continuous.*

PROOF. (See Ahlfors [1], page 65 and 66). Pre and postcomposition of h by affine maps yields a map $\tilde{h} = A \circ h \circ B$ with the same constant M of quasisymmetry. To show h is Hölder continuous at a point p , it suffices to show

\tilde{h} is Hölder continuous at 0 and to assume the intervals I and J contain $[0, 1]$, and that $\tilde{h}(0) = 0$ and $\tilde{h}(1) = 1$. By plugging in $x - t = 0, x = 1/2, x + t = 1$ inequality (2) yields

$$1/(M+1) \leq \tilde{h}(1/2) \leq M/(M+1).$$

Repeated applications of (2) with $x = 1/2^n$ and $t = 1/2^n$ yield

$$1/(M+1)^n \leq \tilde{h}(1/2^n) \leq (M/(M+1))^n.$$

Since \tilde{h} is increasing, this implies that for $1/2^n \leq x \leq 1/2^{n-1}$,

$$\tilde{h}(x) \leq \tilde{h}(1/2^{n-1}) \leq (M/(M+1))^{n-1} = ((M+1)/M) (M/(M+1))^n.$$

But since $x \geq 1/2^n$, the previous inequality implies

$$\tilde{h}(x) \leq ((M+1)/M)x^\alpha, \text{ where } \alpha = \frac{\log((M+1)/M)}{\log 2}.$$

To finish the proof we note that h and \tilde{h} have the same Hölder exponent α . \square

A quasimetric homeomorphism of the unit circle $S^1 = \{e^{i\theta} : \theta \text{ real}\}$ is an orientation preserving homeomorphism of the circle for which there exists a constant M such that

$$\frac{1}{M} \leq \left| \frac{f(e^{i(x+t)}) - f(e^{ix})}{f(e^{ix}) - f(e^{i(x-t)})} \right| \leq M, \quad (3)$$

for all x and all $|t| < \frac{\pi}{2}$. Obviously the restriction of any Möbius transformation preserving the unit disc to the unit circle is quasimetric. Let a finite number of smooth real-valued charts φ_j that cover the circle be given and assume the maps from intervals to intervals defined by $\varphi_j \circ h \circ (\varphi_k)^{-1}$ are quasimetric with constants M_{jk} on the intervals where they are defined. Then h will be quasimetric with a constant M depending on the M_{jk} and the coordinate charts φ_j and φ_k . Conversely, suppose h is quasimetric and φ_j are a finite system of smooth charts whose domains of definition cover the circle. Then each $\varphi_j \circ f \circ (\varphi_k)^{-1}$ is quasimetric on the interval where it is defined. Thus, if when expressed in terms of a finite number of smooth charts that cover the circle h is quasimetric on the intervals for some constant M , then h is quasimetric on the circle for some possibly different constant M_1 .

1.2 The Quasimetric Topology

Now we introduce a topology on the group of orientation preserving homeomorphisms h of a circle that satisfy inequality (3) by specifying a neighborhood basis $V(\epsilon)$ of the identity. By definition h is in $V(\epsilon), \epsilon > 0$, if two conditions are satisfied:

- (a) $\sup\{|h(e^{ix}) - e^{ix}|, |h^{-1}(e^{ix}) - e^{ix}|\} < \epsilon$, and
- (b) inequality (3) is satisfied with $M = 1 + \epsilon$.

This system of neighborhood has the following properties:

- (i) $\bigcap_{n=1}^{\infty} V(1/n) = \{identity\}$,
- (ii) for every $\epsilon > 0$, there exists $\delta > 0$, such that $V(\delta) \circ V(\delta) \subset V(\epsilon)$, and
- (iii) for every $\epsilon > 0$, there exists $\delta > 0$, such that $(V(\delta))^{-1} \subset V(\epsilon)$.

This system of neighborhoods induces a right and a left topology on QS by right and left translation. That is, $V \circ h$ is a right neighborhood of h when V is a neighborhood of the identity. These neighborhoods are precisely those that make right translation maps $h \mapsto h \circ g$ continuous. Similarly, there is the system of left neighborhoods $h \circ V$ of the h , and these make left translation maps $h \mapsto g \circ h$ continuous. However, these properties constitute only part of the structure necessary to make QS a topological group. In the next section we examine this discrepancy in more detail.

1.3 The Symmetric Subgroup

There is a brief, relatively undeveloped, theory of groups that are also Hausdorff topological spaces satisfying axioms (i), (ii), and (iii) above. More details may be found in [35]. We summarize this theory and its application to QS in this section.

DEFINITION. A *topological group* is a group G that is also a Hausdorff topological space and such that the map $(f, g) \mapsto f \circ g^{-1}$ from $G \times G$ to G is continuous.

It turns out that QS is not a topological group because taking inverses is not continuous. However, it does satisfy the axioms for what we call a partial topological group.

DEFINITION. A *partial topological group* is a group with a Hausdorff system of neighborhoods of the identity satisfying (i), (ii) and (iii) above.

As we have seen in section 1.2, at a general point h of the group there are two neighborhood systems. If U runs through the neighborhood system at the identity, then $h \circ U$ and $U \circ h$ run through systems of left and right neighborhoods of h , respectively. The following theorem is proved in [35].

Theorem 1 *The following conditions on a partial topological group are equivalent:*

- i) *it is a topological group with the given neighborhood system of the identity,*

- ii) the left and the right neighborhood systems agree at every point,
- iii) the adjoint map $f \mapsto h \circ f \circ h^{-1}$ is continuous at the identity for every h in the group.

In a general partial topological group the properties of Theorem 1 will not hold. One of the two topologies in a partial topological group will be left translation invariant and the other right translation invariant. The inverse operation interchanges these two topologies.

One can consider those elements h of a partial topological group for which the two neighborhood systems at h agree, that is, for which conjugation by h maps the neighborhood system at the identity isomorphically onto itself. These elements form a closed subgroup: the two topologies agree on this subgroup and give it the structure of a topological group. We call this subgroup the *characteristic topological subgroup*.

If a subset of a partial topological group is invariant under the inverse operation, then it is closed for one topology if, and only if, it is closed for the other. In particular, one may speak without ambiguity of a closed subgroup of a partial topological group.

The next result is elementary.

Theorem 2 *The characteristic topological subgroup of a partial topological group is a closed topological subgroup.*

DEFINITION. A quasisymmetric map h has *vanishing ratio distortion* if there is a function $\epsilon(t)$ with $\epsilon(t)$ converging to zero as t converges to zero, such that inequality (3) is satisfied with M replaced by $1 + \epsilon(t)$.

It turns out that the characteristic topological subgroup of QS comprises precisely those homeomorphisms that have vanishing ratio distortion. We shall call this subgroup the *symmetric subgroup* S . A direct proof that S is a topological group is elementary. Here we prove only the following fact.

Theorem 3 *S is a closed subgroup of QS .*

PROOF. We shall use the following notation. I and J are contiguous intervals, $I = [a, b]$, $J = [b, c]$, and $|I| = b - a$ is the length of I . Let a constant $C > 1$ be given. One first shows that if I and J are contiguous with

$$1/C \leq |I|/|J| \leq C$$

and if g is sufficiently near the identity in the quasisymmetric topology, then

$$\frac{1}{1 + \epsilon} \leq \frac{|g(I)|}{|g(J)|} \cdot \frac{|J|}{|I|} < 1 + \epsilon.$$

Assume h_n is a sequence of elements of S converging in the right QS -topology to h . This means that for sufficiently large n ,

$$\frac{1}{1+\epsilon} \leq \frac{|h \circ h_n^{-1}(I)|}{|h \circ h_n^{-1}(J)|} \cdot \frac{|J|}{|I|} < 1 + \epsilon.$$

Also, assume that for each h_n there is a function $\epsilon_n(t)$ approaching zero as t approaches 0 such that for all contiguous intervals K and L with $1/C \leq |K|/|L| \leq C$,

$$\frac{1}{1+\epsilon_n(|K|)} < \frac{|h_n(K)|}{|h_n(L)|} \cdot \frac{|L|}{|K|} < 1 + \epsilon_n(|K|).$$

Taking the product, we obtain

$$\frac{1}{(1+\epsilon)(1+\epsilon_n(|K|))} \leq \frac{|h \circ h_n^{-1}(I)|}{|h \circ h_n^{-1}(J)|} \cdot \frac{|J|}{|I|} \cdot \frac{|h_n(K)|}{|h_n(L)|} \cdot \frac{|L|}{|K|} < (1+\epsilon)(1+\epsilon_n(|K|)).$$

Since there is a uniform bound on the quasisymmetric norm of h_n , we may assume $1/C \leq \frac{|h_n(K)|}{|h_n(L)|} \leq C$, and thus we may substitute in $I = h_n(K)$ and $J = h_n(L)$. We obtain

$$\frac{1}{(1+\epsilon)(1+\epsilon_n(|K|))} \leq \frac{|h(K)|}{|h(L)|} \cdot \frac{|L|}{|K|} < (1+\epsilon)(1+\epsilon_n(|K|)).$$

For given $\delta > 0$, we can pick n_0 large enough so that $\epsilon_{n_0}(|K|) < \epsilon$ whenever $|K| < \delta$ and $1/C < |K|/|L| < C$. Then

$$\frac{1}{(1+\epsilon)^2} < \frac{|h(K)|}{|h(L)|} \cdot \frac{|L|}{|K|} < (1+\epsilon)^2,$$

and this implies h has vanishing ratio distortion. \square

1.4 Dynamical Systems and Deformations

By definition, *universal Teichmüller space* T is QS factored by the close subgroup of Möbius transformations that preserve the unit disc. It is universal in the naive sense that it contains the deformation spaces of nearly all one-dimensional dynamical systems F that act on the unit circle. When we say this, we have in mind the following two types of dynamical systems. F is either a Fuchsian group acting on the unit circle or a C^2 -homeomorphism acting on the unit circle. In the second situation, it is often useful to assume F has irrational rotation number. The theory is already complicated if F is a diffeomorphism and becomes even more so if F is allowed to have one critical point. Under smooth changes of coordinate, we may assume F maps an interval on the real axis to another interval on the real axis and maps the origin to a point c . We also assume that in suitable smooth coordinates F takes the form of a power law:

$$F(x) = |x|^\alpha \text{sign}(x) + c,$$

for some constant $\alpha > 1$.

The deformation space $T(F)$ (sometimes called the Teichmüller space of F), is defined to be the space of equivalence classes of quasimetric maps $h \in QS$ such that $h \circ F \circ h^{-1}$ is a dynamical system of the same type. Two maps h_0 and h_1 are equivalent if there is a Möbius transformation A such that $A \circ h_0 = h_1$. Thus, it is the set of quasimetric conjugacies to dynamical systems of the same type factored by this equivalence relation.

In the case that F is a Fuchsian group with generators γ_j this means that, for each j , the conjugate $h \circ \gamma_j \circ h^{-1}$ is also a Möbius transformation preserving the unit circle. In the case F is a C^2 homeomorphism possibly with a power law, this means $h \circ F \circ h^{-1}$ is also a C^2 -homeomorphism. If h is itself a Möbius transformation, we consider the dynamical systems generated by F and by $h \circ F \circ h^{-1}$ as not differing in any essential way. For this reason, we view $T(F)$ as a subspace of $QS \bmod PSL(2, \mathbb{R})$. That is, two elements h_1 and h_2 are considered equivalent if there is a Möbius transformation A such that $A \circ h_1 = h_2$.

It turns out that the factor space $T = QS \bmod PSL(2, \mathbb{R})$ carries in a natural way the structure of a complex manifold, as do the subspaces $T(F)$ for many dynamical systems F . Even the statement that $T(F)$ is connected is already significant, and knowledge of geometrical properties of curves which join pairs of points in $T(F)$ can have dynamical consequences.

Here we explain why conjugacies that allow distortion of eigenvalues cannot be smooth: therefore, to obtain interesting conjugacies one must expand into the quasimetric realm.

Lemma 2 *Let F_0 and F_1 be two discrete dynamical systems acting on the real axis, generated by $x \mapsto \gamma_0(x) = \lambda_0 x$ and by $x \mapsto \gamma_1(x) = \lambda_1 x$, respectively, and assume $1 < \lambda_0 < \lambda_1$. Let h be a conjugacy, so that $h \circ \gamma_0 \circ h^{-1} = \gamma_1$. Then h can be at most Hölder continuous with exponent $\alpha = \log \lambda_0 / \log \lambda_1$.*

PROOF. Because $h(\lambda_0^n x) = \lambda_1^n h(x)$, by plugging in $x = 1$ and letting n approach $-\infty$ and ∞ , one sees that h must fix 0 and ∞ . By postcomposition of h with a real dilation, we may assume $h(1) = 1$ and this implies $h(\lambda_0^n) = \lambda_1^n$. But any such map taking these values for arbitrarily large negative values of n cannot satisfy an inequality of the form $|h(x)| \leq C|x|^\alpha$ unless $\alpha \leq \log \lambda_0 / \log \lambda_1$. \square

1.5 Tangent Spaces to QS and S

In this section we identify the circle with the extended real line $\overline{\mathbb{R}}$. By postcomposing with a Möbius transformation we may assume any homeomorphism representing an element of T fixes infinity. Consider a smooth curve h_s of homeomorphisms in QS parameterized by s and passing through the identity at $s = 0$. We may assume each homeomorphism h_s fixes infinity. By smooth we shall mean that

$$h_s(x) = x + sV(x) + o(s), \quad (4)$$

where the distance measured in the quasisymmetric norm from the identity to h_s is less than or equal to a constant times s . In particular,

$$\frac{1}{1 + Cs} \leq \frac{h_s(x + t) - h_s(x)}{h_s(x) - h_s(x - t)} \leq 1 + Cs.$$

By substituting (4) into this formula we arrive at the following condition on the continuous function V :

$$|V(x + t) - 2V(x) + V(x - t)| = O(t). \quad (5)$$

If h_s is a smooth curve in the symmetric subspace S , then

$$|V(x + t) - 2V(x) + V(x - t)| = o(t). \quad (6)$$

We will call (5) and (6), respectively, the big and little Zygmund conditions. Since V is to be regarded as the tangent vector to the one-parameter family of homeomorphisms h_s , $V(x) \frac{\partial}{\partial x}$ is a vector field.

If instead we consider the mappings h_s as acting on the unit circle $|z| = 1$ then the condition that the vector field W point in a direction tangent to the unit circle is that

$$\tilde{W}(x) = W(e^{ix})/ie^{ix} \quad (7)$$

be real-valued. The boundedness conditions on QS and S correspond to the conditions that the continuous, periodic function \tilde{W} satisfy (5) and (6). We denote the spaces of continuous vector fields satisfying these conditions by Z and Z_0 , respectively.

A simple example of a tangent vector in Z_0 is generated by a curve of Möbius transformations preserving the unit circle and passing through the identity. Such a curve has a tangent vector of the form

$$W(z) \frac{\partial}{\partial z} = (\alpha z^2 + \beta z + \gamma) \frac{\partial}{\partial z},$$

where α , β , and γ are constants which make $W(z)$ real-valued along $z\bar{z} = 1$. We call such tangent vectors trivial. Thus the quadratic polynomials which satisfy this reality condition define the tangent vectors to trivial curves of homeomorphisms.

We will show that any tangent vector satisfying the big Zygmund condition is the tangent vector to a smooth curve in QS passing through the identity, and correspondingly any tangent vector satisfying the little Zygmund condition is the tangent vector to a smooth curve in S .

DEFINITION. Let \mathcal{Z} and \mathcal{Z}_0 be the spaces Z and Z_0 factored by the quadratic polynomials.

Eventually in section 2.3 we shall identify a Banach space \mathcal{A} such that the Banach dual of \mathcal{Z}_0 is isomorphic to \mathcal{A} and the Banach dual of \mathcal{A} is isomorphic to \mathcal{Z} . In particular, $\mathcal{Z}_0^{**} \cong \mathcal{Z}$.

Let Q be any quadruple of points a, b, c and d arranged in counter-clockwise order on the unit circle or in increasing order on the real axis and define the cross ratio $cr(Q)$ by

$$cr(Q) = \frac{(d-c)(b-a)}{(c-b)(a-d)}. \quad (8)$$

Recall that $cr(Q)$ is Möbius invariant in the sense that $cr(A(Q)) = cr(Q)$ for any Möbius transformation A . In consequence we may define a norm $\| \cdot \|_{cr}$ on vector fields which is Möbius invariant in the sense that

$$\|W\|_{cr} = \left\| \frac{W \circ A}{A'} \right\|_{cr},$$

for every Möbius transformation A .

Define $W[a, b, c, d]$ to be the alternating sum

$$\frac{W(d) - W(c)}{d - c} - \frac{W(c) - W(b)}{c - b} + \frac{W(b) - W(a)}{b - a} - \frac{W(a) - W(d)}{a - d}.$$

For a given quadruple Q the term $cr(Q)\rho(cr(Q))W[a, b, c, d]$ measures the velocity of the cross-ratio (8) with respect to the Poincaré metric $\rho(z)|dz|$ on the sphere punctured at 0, 1 and ∞ when each of the points a, b, c and d move with complex velocities $W(a), W(b), W(c)$ and $W(d)$, respectively. The *infinitesimal cross-ratio norm* is defined for the space \mathcal{Z} by

$$\|W\|_{cr} = \sup_Q |cr(Q)\rho(cr(Q))W[a, b, c, d]|. \quad (9)$$

Note that $\|W\|_{cr} = 0$ if, and only if, W is a quadratic polynomial. Furthermore, if Q has the form $Q = (-\infty, x-t, x, x+t)$ then $cr(Q) = -1$. If in addition we assume $|W(z)| = o(|z|^2)$, which is tantamount to the assumption that $W(z)\frac{\partial}{\partial z}$ vanishes at infinity, then the alternating sum $W[a, b, c, d]$ is equal to

$$-\frac{W(x+t) - 2W(x) + W(x-t)}{t}.$$

1.6 The Hilbert Transform and Almost Complex Structure

If the vector field $\tilde{V}(z)\frac{\partial}{\partial z}$ is continuous and real-valued on the circle, then the function $V(x) = \tilde{V}(e^{ix})/ie^{ix}$ is continuous, real-valued and periodic on the real axis. Define a function $W(x)$ by the formula

$$W(x) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|y-x| \geq \epsilon} V(y) \cot\left(\frac{y-x}{2}\right) dy, \quad (10)$$

where the integral is taken over values of y for which $-\pi \leq y \leq \pi$ and $|y - x \pm 2\pi n| \geq \epsilon$ for all integers n . Transporting W back to the unit circle by the formula

$$\tilde{W}(e^{ix}) = W(x)ie^{ix},$$

one obtains a complex-valued function \tilde{W} defined on the circle for which

$$\tilde{W}(z) \frac{\partial}{\partial z}$$

is again real-valued. By this process \tilde{V} is transformed to $J\tilde{V} = \tilde{W}$, another field of vectors on the unit circle whose directions are tangent to the circle.

This rule defines an operator J called the *Hilbert transform*; it extends to a bounded operator for many different smoothness classes. For example,

$$\|W\|_p \leq C_p \|V\|_p$$

for $p \geq 2$, where

$$\|V\|_p^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} |V(x)|^p dx.$$

More important to us are the following properties of J :

1. the Zygmund classes Z and Z_0 are preserved by J ,
2. J is anti-involutory in the sense that $J^2 = -I$, and
3. $J(\sin kx) = \cos kx$ and $J(\cos kx) = -\sin kx$.

Proofs of all of these statements are greatly simplified by considering different possible extensions of V to the complex plane, as we shall see in chapter 2.

The anti-involutory property of J yields an almost complex structure on \mathcal{Z} the tangent space to universal Teichmüller space. In fact, whenever an anti-involutory automorphism J of a vector space X over \mathbb{R} is given, X becomes a vector space over \mathbb{C} by defining, for every v in X ,

$$(a + ib)v = av + bJ(v).$$

The reader should check that $((a_1 + ib_1)(a_2 + ib_2))v = (a_1 + ib_1)((a_2 + ib_2)v)$.

Instead of using the exponential map $z = e^{ix} \mapsto x$ as a real-valued chart for the unit circle, one can use the stereographic map $z \mapsto u$ from the circle to $\overline{\mathbb{R}}$ where

$$u = U(z) = \frac{z + i}{iz + 1}. \quad (11)$$

This map sends the four points $1, i, -1, -i$ on the unit circle to the four points $1, \infty, -1, 0$, respectively, on the real axis. The real-valued vector field $\tilde{V}(z) \frac{\partial}{\partial z}$ on the circle $\{z : |z| = 1\}$ is related to the vector field $\hat{V}(u)$ defined for u on the real axis by

$$\tilde{V}(z) = \hat{V}(u) \left(\frac{-2}{(u + i)^2} \right).$$

Since we assume \tilde{V} is continuous, and in particular bounded on the circle, that implies at most quadratic growth of \hat{V} near ∞ . That is

$$\hat{V}(u) = O(|u|^2)$$

as $|u| \rightarrow \infty$.

In the special case when $\tilde{V}(z)\frac{\partial}{\partial z} = (c_0 + c_1 z + c_2 z^2)\frac{\partial}{\partial z}$ is a real-valued Möbius vector field, then c_1 is pure imaginary, $c_2 = -\overline{c_0}$,

$$\hat{V}(u) = \left(c_0 + c_1 \left(\frac{i(u-i)}{(u+i)} \right) + c_2 \left(\frac{i(u-i)}{(u+i)} \right)^2 \right) \left(\frac{-2}{(u+i)^2} \right)$$

is a quadratic polynomial in u , and

$$V(x) = \frac{(c_0 + c_1 e^{ix} - \overline{c_0} e^{2ix})}{ie^{ix}} = \frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x.$$

Here, a_0, a_1 , and b_1 are real and $a_0 = 2 \operatorname{Im} c_1, a_1 = 2 \operatorname{Im} c_2, b_1 = 2 \operatorname{Re} c_2$.

It will turn out that the quadratic polynomials are preserved by the Hilbert transform and so J is well-defined on the quotient space

$$\mathcal{Z} = \{\tilde{V} \in Z\} / \{\text{quadratic polynomials}\}. \quad (12)$$

In section 2.6, when we use complex methods to deal with Hilbert transforms, we will find it useful to map the interior of the circle to the upper half-plane by the stereographic map $u = U(z)$ given in (11) and then compute the Hilbert transform in the upper half-plane. When this is done, it must be remembered that the function \hat{V} is permitted to have at most quadratic growth near infinity.

1.7 Scales and Trigonometric Approximation

If a vector field $\tilde{V}(z)\frac{\partial}{\partial z}$, on the unit circle is given by a finite sum of the form

$$\tilde{V}_n(z) = \sum_{k=-n}^n c_k z^k, \quad (13)$$

and is real-valued, then $c_{n+2} = -\overline{c_{-n}}$. The corresponding function $V_n(x) = \tilde{V}_n(e^{ix})/ie^{ix}$, is the trigonometric polynomial

$$a_0/2 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$$

of degree n , where $a_k = 2 \operatorname{Im} c_{k+1}$ and $b_k = 2 \operatorname{Re} c_{k+1}$. We may think of the trigonometric polynomial V_n with

$$\|V_n\|_\infty = M$$

as a typical vector field having a definite oscillation down to intervals whose length is as small as $\frac{1}{Mn}$. That is, if $V(x) = 1$ and $0 < t < \frac{1}{Mn}$, then $V(x+t) > 0$. This is because of the mean value theorem and the following lemma due to Bernstein [5].

Lemma 3 *If $V_n(x)$ is a trigonometric polynomial of degree n , then*

$$\left\| \frac{d}{dx} V_n(x) \right\|_{\infty} \leq n \|V_n(x)\|_{\infty}.$$

PROOF. We follow the proof given in [55], page 39. To begin, assume there is a trigonometric polynomial V_n with $\|V'_n\| = nL$ and $L > \|V_n\|$. Thus at some point x_0 , $|V'_n(x_0)| = nL$, and we can assume that $V'_n(x_0) = nL$. Since V'_n is a maximum at x_0 , $V''(x_0) = 0$.

Consider the trigonometric polynomial

$$T_n(x) = L \sin n(x - x_0) - V_n(x)$$

of degree n . In the interval $[x_0, x_0 + 2\pi)$ there are $2n$ points where $\sin n(x - x_0)$ takes the values ± 1 , and between any two of these points the polynomial T_n takes values of opposite sign. Hence T_n has $2n$ different zeros in this interval, and so

$$T'_n(x) = nL \cos n(x - x_0) - V'_n(x)$$

also has $2n$ different zeros. One of these zeros is x_0 , since

$$T'_n(x_0) = nL - V'_n(x_0).$$

Also,

$$T''_n(x) = -n^2 L \sin n(x - x_0) - V''_n(x)$$

vanishes at $x = x_0$. Moreover, T''_n has $2n$ zeroes between the zeros of T'_n . Thus T''_n has at least $2n + 1$ zeros in this interval, and since it is a trigonometric polynomial of degree n it must be identically zero. Thus T'_n is constant, but since $T'_n(x_0) = 0$, this implies T_n is constant. But this contradicts the statement that T_n changes sign and we conclude that the original assumption could not be correct, that is, we have $L \leq \|V_n\|_{\infty}$, which means $\|V'_n\|_{\infty} \leq n \|V_n\|_{\infty}$. \square

Assume V and W are continuous functions of period 2π . An inequality of the form $\|V(x) - W(x)\|_{\infty} < 1/2^n$ for large n implies that the graph of $V(x)$ closely resembles the graph of $W(x)$. Now consider $M_{k,I}V(x) = \frac{1}{2^k}V(2^k x)$, where x lies in some interval I of length $2\pi/2^k$. Then $M_{k,I}$ is a magnification operator of degree k , magnifying the graph of V over the interval I by the same factor in both the domain and range.

In fractal geometry, one considers graphs that have roughly the same shape no matter how much they are magnified. Thus, suppose that we go to some fine scale $M_{k,I}V$. Then the picture of the graph seen at this scale should roughly resemble the picture of the graph of $M_{n,J}$ if n is any number larger than k and J is any interval of size $2\pi/2^n$.

A general trigonometric polynomial does not possess this property. Suppose a polynomial V_{2^n} of degree 2^n is magnified by the operator $M_{k,I}$ of degree k . If k is larger than n , then because of Bernstein's inequality, one does not see any oscillation in the graph of $M_{k,I}V_n$. This observation motivates the following theorem due to Zygmund and Jackson, [77], [43], [55].

Theorem 4 Suppose $V(x)$ is a continuous, periodic function defined on the real axis. Then V is in the Zygmund class Z defined by

$$\left| \frac{V(x+t) + V(x-t) - 2V(x)}{t} \right| \leq C$$

if, and only if, there exists a constant C' such that for every positive integer n there is a trigonometric polynomial V_n of degree at most n , such that

$$\|V - V_n\|_\infty \leq \frac{C'}{n}.$$

Moreover, the number C' can be estimated purely in terms of C and vice versa.

PROOF. We begin by proving that if such trigonometric approximations are possible for every n , then V is in the Zygmund class. For each integer of the form 2^k , let V_{2^k} be a trigonometric polynomial of degree 2^k such that

$$\|V - V_{2^k}\|_\infty \leq \frac{C}{2^k}. \quad (14)$$

Now select n so that $\frac{1}{2^{n+1}} \leq t \leq \frac{1}{2^n}$, and write V in the form $V = W_1 + W_2$ where $W_1 = V - V_{2^n}$ and $W_2 = V_{2^n}$. In general, define the difference operator Δ_t by

$$\Delta_t G(x) = G(x+t) - G(x).$$

Then

$$\Delta_t^2 G(x) = \Delta_t(\Delta_t(G))(x) = G(x+2t) - 2G(x+t) + G(x).$$

From the hypothesis,

$$|\Delta_t^2 W_1(x)| \leq \frac{8C}{2^{n+1}} \leq 8Ct. \quad (15)$$

Putting $V_0 = 0$, we may rewrite W_2 as a sum over scales:

$$W_2(x) = V_1 - V_0 + \sum_{k=1}^n (V_{2^k} - V_{2^{k-1}}). \quad (16)$$

Each term $V_{2^k} - V_{2^{k-1}}$ has norm bounded by

$$|V_{2^k} - V| + |V - V_{2^{k-1}}| \leq \frac{3C}{2^k},$$

and is a trigonometric polynomial of degree less than or equal to 2^k . So by Lemma 3, the second derivative of $V_{2^k} - V_{2^{k-1}}$ is at most $3C2^k$. Thus, by the second mean value theorem

$$|\Delta_t^2 (V_{2^k} - V_{2^{k-1}})| \leq 3C2^k t^2.$$

By using equation (16), we obtain

$$|\Delta_t^2 W_2| \leq \sum_{k=1}^n 3C2^k t^2 \leq \sum_{k=1}^n \frac{3C2^k}{2^{2n}} = \frac{3C2^{n+1}}{2^{2n}} = \frac{6C}{2^n} \leq 12Ct. \quad (17)$$

Putting inequalities (15) and (17) together, we obtain $|\Delta_t^2 V| \leq 20Ct$ and this proves the first half of the theorem.

To prove the other half, for every n we must construct a trigonometric polynomial V_n of degree n that approximates V in the sup-norm to within C/n . Let K_n be the Jackson kernel defined by $K_n = \sigma_{2n-1} - 2\sigma_n$, where

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} = \frac{1}{2\pi n} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^2$$

is the Fejér kernel and

$$s_n = \frac{\sin(2n+1)\frac{t}{2}}{2 \sin \frac{t}{2}}.$$

By convolution of V with the Jackson kernel K_n , one gets a trigonometric polynomial of degree $2n-1$ that approximates V to within $O(1/n)$ in the sup norm. For details of this proof we refer to [55], pages 55-56, or to [77]. \square

Theorem 5 *The Zygmund spaces Z and Z_0 are invariant under the Hilbert transform J .*

PROOF. Since a much easier proof of the same result is given in section 2.6, here we only outline the argument given by Zygmund in [77]. Begin by using a result of Favard [25]: if $|g'| \leq M$ then $|Jg - J\sigma_n(g)| \leq A/n$. Then employ the Zygmund-Jackson theorem. Let V be in Z and V_n be a trigonometric polynomial of degree n with $|V - V_n| \leq \frac{C}{n}$. Let $G' = V$ and $T'_n = V_n$. Then $|(G - T_n)'| \leq \frac{C}{n}$ and therefore $|J(G - T_n) - J\sigma_n(G - T_n)| \leq \frac{AC}{n^2}$. Thus JG is approximable in the sup norm to within $\frac{AC}{n^2}$ by a trigonometric polynomial of degree n . This implies that JV is approximable to within $\frac{AC}{n}$ by a trigonometric polynomial of degree n , therefore JV is in the Zygmund class.

The proof for the class Z_0 is similar. \square

1.8 Automorphisms of Teichmüller Space

Given any quasisymmetric homeomorphism f of S^1 , the map $\rho_f([h]) = [h \circ f^{-1}]$ is a bicontinuous self-map of $T = QS \bmod PSL(2, \mathbb{R})$. Moreover, ρ_f preserves the almost complex structure. We call biholomorphic automorphisms of T of this form *geometric* automorphisms.

An *almost complex structure* on a real Banach manifold M is a smoothly varying family of automorphisms $J_x, x \in M$, of each fiber of the tangent bundle such that $J_x^2 = -I$. A diffeomorphism F of M is *almost complex* if $J_{F(x)}(F'_x(v)) = F'_x(J_x(v))$.

Theorem 6 *Any almost complex automorphism F of T is geometric. That is, given a diffeomorphism F of*

$$QS \bmod PSL(2, \mathbb{R})$$

whose derivative commutes with the almost complex structure, there exists a quasisymmetric map f such that $F = \rho_f$.

An outline of the proof of this theorem is given in the last section of this paper.

2 Complex Analysis

2.1 Quasiconformal Extensions

Roughly speaking, a homeomorphism of \mathbb{R}^n is quasiconformal if it distorts standard shapes by a bounded amount, see [54], [63], [33]. When $n \geq 2$, it turns out that quasiconformal maps are differentiable almost everywhere and the distortion of shape can be measured infinitesimally. An observation of central importance for the deformation theory of one-dimensional dynamical systems is that this statement is not true when $n = 1$. That is, quasisymmetric maps may not be differentiable anywhere.

In any case, the measurement of quasiconformal distortion at a point z for a mapping f when $n = 2$ is by means of a quantity called the *local dilatation* $K_z(f)$:

$$K_z(f) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}. \quad (18)$$

Any quasisymmetric homeomorphism h of the real axis extends to a quasiconformal self-mapping of the upper half-plane. This pivotal result was first proved by Ahlfors and Beurling [4]. The formula given in [1] for such an extension of h is $H_1(z) = F(z) + iG_1(z)$, where

$$\begin{aligned} F(x + iy) &= \frac{1}{2y} \int_{x-y}^{x+y} h(t) dt \\ G_1(x + iy) &= \frac{1}{2y} \left\{ \int_x^{x+y} h(t) dt - \int_{x-y}^y h(t) dt \right\}. \end{aligned} \quad (19)$$

This formula does not extend the identity by the identity. In particular, for $h(x) = x$, the extension $H_1(z) = x + \frac{1}{2}iy$. It is therefore convenient to multiply the expression for G_1 by a factor two. That is, we put

$$G(x + iy) = \frac{1}{y} \left\{ \int_x^{x+y} h(t) dt - \int_{x-y}^y h(t) dt \right\}. \quad (20)$$

Although $H = F + iG$ differs from H_1 , it still yields a quasiconformal extension $ex(h)$ of h , but with the additional property that the identity is extended by the identity. It is useful to view H as an extension to the whole plane by stipulating that $H(\bar{z}) = \overline{H}(z)$. The reader should check that this extension process is natural for real affine transformations in the sense that if $A(z) = c_1z + c_2$ and $B(z) = c_3z + c_4$ where c_1, \dots, c_4 are real, then

$$ex(A \circ h \circ B) = A \circ ex(h) \circ B.$$

Hence, if we assume

$$h(x) + 1 = h(x + 1), \quad (21)$$

then $H(z + 1) = H(z) + 1$.

It is important to note that a lift of a self-homeomorphism of the circle by the universal covering $x \mapsto e^{2\pi ix}$ yields a homeomorphism h satisfying (21), and conversely, if a homeomorphism of the real-axis satisfies (21) then it projects to a homeomorphism of the circle. Moreover, the covering $x \mapsto e^{2\pi ix}$ extends to the covering $z \mapsto e^{2\pi iz}$ of the punctured unit disc $\mathbb{D}^* = \mathbb{D} - \{0\}$ by the upper half-plane. The extension of h to the disc punctured at 0 is a quasiconformal map preserving 0, and therefore if we stipulate that the extension preserves 0, it becomes a quasiconformal extension to the entire disc. This method of extension also has the following *asymptotic property*:-

Assume that $h(0) = 0$, $h(x + 1) = h(x) + 1$, h is quasisymmetric and

$$\frac{1}{1 + \epsilon} \leq \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq 1 + \epsilon$$

for $|t| < \epsilon$, with δ sufficiently small. Then if $|\operatorname{Im} z| < \delta$, the dilatation K_z of $ex(h)$ at z satisfies $K_z < 1 + \epsilon'$, where ϵ' converges to zero as ϵ converges to zero.

2.2 Teichmüller's Metric

The Teichmüller distance between two points $[h_1]$ and $[h_2]$ in $QS \bmod PSL(2, \mathbb{R})$ is defined to be

$$d([h_1], [h_2]) = \frac{1}{2} \log K_0(h_2 \circ (h_1)^{-1}), \quad (22)$$

where

$$K_0(h) = \inf \{K(\tilde{h}) : \text{where } \tilde{h} \text{ is any quasiconformal extension of } h\}.$$

As a consequence of basic properties of quasiconformal mappings, $d([id], [h])$ is always realized by an extremal mapping \tilde{h} which is an extension of h . To see this one can assume that h fixes three points on the real axis, say 0, 1 and ∞ , and then select extensions \tilde{h}_n of h such that $\frac{1}{2} \log K(\tilde{h}_n) < \frac{1}{2} \log K_0([h]) + \frac{1}{n}$. Since the mappings \tilde{h}_n are normalized and have uniformly bounded dilatation, they are equicontinuous. Therefore there is a subsequence of \tilde{h}_n that converges uniformly in the spherical metric to some self-mapping of the upper half-plane \tilde{h}_0 . Since each \tilde{h}_n coincides with h at every point of the real axis, so does \tilde{h}_0 . Moreover, the maximal dilatation of \tilde{h}_0 must be less than $K([h]) + \frac{1}{n}$ for every positive integer n . On the other hand $K(\tilde{h}_0)$ cannot be less than $K_0([h])$ because by definition $K_0([h])$ is the infimum of the dilatations of all possible extensions of h . We conclude that every mapping h of the real axis has an *extremal* quasiconformal extension \tilde{h}_0 to the upper half-plane, that is, an extension for which

$$K_0([h]) = K(\tilde{h}_0).$$

It will turn out that certain quasiconformal mappings h have many extremal extensions and thus we do not expect to find a general formula that yields an extremal extension. In particular, the Beurling-Ahlfors extension formula given in the previous section will almost never yield an extremal extension.

In formula (22) the Teichmüller distance is seen as the solution to an infimum problem. It turns out that it is also the solution to a supremum problem. Consider the vector space $\mathcal{R}(\mathbb{H})$ of integrable holomorphic quadratic differentials $\varphi(z)dz^2$ in the upper half-plane \mathbb{H} with only a finite number of poles on the real axis and for which $\varphi(z)dz^2$ is real-valued on the real axis. Any element $\varphi(z)dz^2$ of $\mathcal{R}(\mathbb{H})$ has the form

$$\varphi(z)dz^2 = \frac{p(z)dz^2}{(z-x_1)\cdots(z-x_n)}, \quad (23)$$

where x_1, \dots, x_n are distinct points on the real axis and $p(z)$ is a polynomial of degree less than or equal to $n-3$ with real coefficients. Such a quadratic differential determines a decomposition of \mathbb{H} into a finite number of strips, S_1, \dots, S_k , where $k \leq n-3$. The interior of each strip is swept out by horizontal trajectories of this quadratic differential, that is, parameterized curves $\alpha(t)$ along which $\varphi(\alpha(t))\alpha'(t)^2 > 0$. A choice of coordinate

$$\zeta = \pm \int \sqrt{\varphi(z)}dz + (const)$$

can be made so that $z \mapsto \zeta$ maps the j -th strip to a rectangle R_j and takes the horizontal trajectories $\alpha(t)$ to horizontal line segments that join the left side of the rectangle to its right side. Let a_j and b_j be the width and height of the rectangle R_j measured in the parameter ζ . Then

$$\|\varphi\| = \int_{\mathbb{H}} |\varphi(z)|dxdy = \sum_{j=1}^k a_j b_j.$$

That is $\|\varphi\|$ is equal to the sum of the areas of these rectangles. Moreover, if β is any arc in \mathbb{H} with endpoints on $\partial\mathbb{H}$ transversal to the horizontal trajectories of φ , then we can assign to it a *height* $ht_\varphi(\beta)$ given by

$$ht_\varphi(\beta) = \int_\beta \text{Im}(\sqrt{\varphi(z)}dz),$$

equal to the sum of the heights of the rectangles corresponding to the strips S_j crossed by β .

Let I_j be the intervals on $\partial\mathbb{H}$ whose endpoints are successive pairs from the (ordered) sequence of points x_1, \dots, x_k and assume that the endpoints of the arc $\beta \subset \mathbb{H}$ lie on the intervals I_{j_1} and I_{j_2} . If β' is another arc transverse to the horizontal trajectories of $\varphi(z)dz^2$ with endpoints on the same intervals I_{j_1} and I_{j_2} , then $ht_\varphi(\beta) = ht_\varphi(\beta')$, and so the height function ht_φ is a nonnegative function defined on all possible pairs of intervals I_{j_1} and I_{j_2} taken from the set I_1, \dots, I_k .

A sense-preserving selfmap h of $\partial\mathbb{H}$ takes the points x_j to points $x'_j = h(x_j)$ and thus determines a new height function defined on pairs of intervals $I'_{j_1} = h(I_{j_1})$ and $I'_{j_2} = h(I_{j_2})$. From the theorem of Hubbard and Masur (in [41]; see also [28]) there is a unique quadratic differential of the form

$$\psi(z)dz^2 = \frac{q(z)dz^2}{(z - x'_1) \dots (z - x'_n)},$$

such that $q(z)$ has real coefficients and degree $\leq n - 3$ and the heights of ψ for the interval pairs I'_{j_1} and I'_{j_2} are equal to the heights of φ for the corresponding interval pairs I_{j_1} and I_{j_2} . Moreover, ψ is unique among all continuous integrable quadratic differentials on $\overline{\mathbb{C}} \setminus \{x'_1, \dots, x'_n\}$, real-valued on the real axis, with heights between pairs of intervals I'_{j_1} and I'_{j_2} greater than or equal to the corresponding heights of φ on intervals I_{j_1} and I_{j_2} and with $\|\psi\|$ as small as possible.

If h has a quasiconformal extension \tilde{h} with dilatation K_0 , then $\|\psi\| \leq K_0\|\varphi\|$, and one obtains the following expression for K_0 :

$$K_0 = \sup \frac{\|\psi\|}{\|\varphi\|}, \quad (24)$$

where the supremum is taken over all non-zero quadratic differentials φ of the form (23) and ψ is the quadratic differential with simple poles at the points $x'_j = h(x_j)$, $1 \leq j \leq n$, and with the same corresponding heights with respect to these points that φ has with respect to the points x_j , $1 \leq j \leq n$.

2.3 Quadratic Differentials

Let $\mathcal{A} = \mathcal{A}(\Omega)$ be the Banach space of integrable functions $\varphi(z)$ holomorphic in Ω where $\Omega = \mathbb{H}$ or $\Omega = \Delta = \{z : |z| < 1\}$ with norm

$$\|\varphi\| = \iint_{\Omega} |\varphi(z)| dx dy < \infty.$$

In this section we introduce a pairing between \mathcal{A} and \mathcal{Z} and show that $\mathcal{A}^* \cong \mathcal{Z}$ and that $(\mathcal{Z}_0)^* \cong \mathcal{A}$. By \mathcal{Z} we mean the vector fields V defined on $\partial\Omega$ such that $V(z)\frac{\partial}{\partial z}$ is real-valued on $\partial\Omega$, and such that $\|V\|_{\mathcal{Z}} < \infty$. Since there is a Möbius transformation transforming \mathbb{H} onto Δ and since the statements we prove will be invariant under pull-back by Möbius transformations, we can work interchangeably with either \mathbb{H} or with Δ .

Our first step is to prove a special case of Bers' approximation theorem [6], [2]. Let $\mathcal{R}(\mathbb{H})$ be the space of finite linear combinations of the form

$$\lambda_1 \varphi_{x_1}(z) + \dots + \lambda_n \varphi_{x_n}(z), \quad (25)$$

where x_1, \dots, x_n and $\lambda_1, \dots, \lambda_n$ are real numbers and

$$\varphi_x(z) = \frac{x(x-1)}{z(z-1)(z-x)}.$$

Theorem 7 \mathcal{R} is dense in \mathcal{A} .

PROOF. A similar and much deeper result is true if \mathbb{H} is replaced by any plane domain, [2], [6].

Let L be any linear functional on the Banach space \mathcal{A} that annihilates \mathcal{R} . To show that \mathcal{R} is dense in \mathcal{A} it is sufficient to show that L annihilates \mathcal{A} . By the Hahn-Banach and Riesz representation theorems, there exists a bounded measurable function μ defined in \mathbb{H} so that

$$L(\varphi) = \text{real part of } \left(\iint_{\mathbb{H}} \mu(z) \varphi(z) dx dy \right).$$

If we extend $\varphi(z)$ and $\mu(z)$ to the lower half-plane by the rules $\overline{\varphi(z)} = \varphi(\bar{z})$ and $\overline{\mu(z)} = \mu(\bar{z})$, then we can write the formula for L as

$$L(\varphi) = \frac{1}{2} \iint_{\mathbb{C}} \mu(z) \varphi(z) dx dy.$$

The assumption that L annihilates \mathcal{R} implies that

$$V(z) = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta = 0 \quad (26)$$

whenever z is a real number. One shows that V has the following properties:

1. $\bar{\partial}V = \mu$ in the sense of distributions,
2. $|V(z)| = O(|z| \log |z|)$ as $z \rightarrow \infty$, and
3. $V(z)$ has an $|\epsilon \log \epsilon|$ -modulus of continuity, that is to say, given $R > 0$, there exists a C such that for every z_1 and z_2 with $|z_1|$ and $|z_2| < R$ and with $|z_1 - z_2| < 1/2$,

$$|V(z_1) - V(z_2)| \leq C|z_1 - z_2| \log(1/|z_1 - z_2|).$$

Let $D_{\epsilon, R}$ be a semi-disc in the upper half-plane with diameter of length $2R$ along the line $y = \epsilon$ and with midpoint on the y -axis. The curved part of the boundary of D_{ϵ} is parameterized by the curve $z = i\epsilon + Re^{i\theta}$, $0 \leq \theta \leq \pi$. Assume further that φ is continuous on the real axis and $\varphi(z) = O(|z|^{-4})$ as $z \rightarrow \infty$. Then the subspace of $A(\mathbb{H})$ comprising those φ with these properties is dense in $A(\mathbb{H})$. Since φ is integrable and μ is bounded,

$$\iint_{\mathbb{H}} \mu \varphi = \lim \iint_{D_{\epsilon, R}} \mu d\xi d\eta,$$

where the limit is taken both as $\epsilon \rightarrow 0$ and as $R \rightarrow \infty$. On the other hand, from Green's formula,

$$\iint_{D_{\epsilon, R}} \mu d\xi d\eta = \int_{\partial D_{\epsilon, R}} V(\zeta) \varphi(\zeta) d\zeta.$$

Because $V(z)$ is identically zero when $z \in \mathbb{R}$, if we first take the limit in this line integral as $\epsilon \rightarrow 0$ we obtain

$$\int_0^\pi V(Re^{i\theta})\varphi(Re^{i\theta})Rie^{i\theta}d\theta = 0.$$

Because of the vanishing condition on φ ,

$$\int_0^\pi V(Re^{i\theta})\varphi(Re^{i\theta})Rie^{i\theta}d\theta$$

is dominated by a constant times $(\log R)/R$ and thus vanishes as $R \rightarrow \infty$. \square

We are now ready to introduce the pairing between an element V in \mathcal{Z} and φ in \mathcal{A} . Given V in \mathcal{Z} we select any extension \tilde{V} of V to the upper half-plane with the properties that $\partial\tilde{V} = \mu$ is essentially bounded and that $|V(z)| = O(|z|^2)$. Then we define

$$(V, \varphi) = \operatorname{Re} \left(\iint_{\mathbb{H}} \mu \varphi \right). \quad (27)$$

We must show first that any V in \mathcal{Z} has such an extension and second that if a different extension is taken, the integration (27) yields the same result. \tilde{V} can be defined by the Beurling-Ahlfors' formula (19) applied to the vector field V :

$$\begin{aligned} \operatorname{Re}(\tilde{V}(x + iy)) &= \frac{1}{2y} \int_{x-y}^{x+y} V(t)dt \quad \text{and} \\ \operatorname{Im}(\tilde{V}(x + iy)) &= \frac{1}{y} \left(\int_x^{x+y} V(t)dt - \int_{x-y}^x V(t)dt \right). \end{aligned}$$

We leave it to the reader to verify that \tilde{V} has the appropriate growth rate and that the Zygmund condition implies

$$\bar{\partial}\tilde{V} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \tilde{V}$$

is bounded. To show that the right hand side of (27) depends only on the values of V on the real axis we first note that from Theorem 7 it suffices to show the right hand side of (27) depends only on the values of V on the real axis when φ has the special form (25). In that case, if we assume V vanishes at 0 and 1 and has growth rate $o(|z|^2)$, then by Green's formula,

$$(V, \varphi) = \frac{\pi}{2} \sum_j \lambda_j V(x_j). \quad (28)$$

Theorem 8 *The pairing (27) induces an isomorphism between \mathcal{Z} and \mathcal{A}^* .*

PROOF. From the preceding discussion we have seen how an element L of \mathcal{A}^* determines by the correspondence an element V in \mathcal{Z} . Conversely, because of the residue formula (28), the extension formula and because \mathcal{R} is dense in \mathcal{A} , any element V of \mathcal{Z} determines by this correspondence an element of \mathcal{A}^* . Note that $\|V\|_{cr} = 0$ is equivalent to the condition that $V(z) = a_0 + a_1z + a_2z^2$. Since by definition elements

$$\varphi(z) = \sum_{j=1}^n \frac{\lambda_j}{z - x_j}$$

of \mathcal{R} satisfy $|\varphi(x)| = O(|z|^{-4})$ as $z \rightarrow \infty$, they also satisfy $\sum \lambda_j = 0$, $\sum_j \lambda_j x_j = 0$ and $\sum_j \lambda_j x_j^2 = 0$. One therefore sees that any quadratic polynomial vector field $V(x) \frac{\partial}{\partial x}$ annihilates all elements of \mathcal{R} , and so also annihilates \mathcal{A} since \mathcal{R} is dense in \mathcal{A} . \square

Because \mathcal{A}^* is isomorphic to \mathcal{Z} , the norm on \mathcal{Z} dual to the norm on \mathcal{A} is equivalent to $\|\cdot\|_{cr}$. We call this norm the *infinitesimal Teichmüller norm* and denote it by $\|\cdot\|_T$. It is given by either of the following formulas:

$$\begin{aligned} \|V\|_T &= \sup \left\{ \left| \iint_{\mathbb{H}} \varphi \bar{\partial} \tilde{V} dx dy \right| : \varphi \in \mathcal{A} \text{ with } \|\varphi\| = 1 \right\} \\ &= \inf \{ \|\bar{\partial} \tilde{V}\|_{\infty} : \text{where } \tilde{V} \text{ is any extension of } V. \} \end{aligned}$$

DEFINITION. We say a sequence φ_n in \mathcal{A} is *degenerating* if there is a constant $C > 1$ such that $C^{-1} \leq \|\varphi_n\| \leq C$ and $\varphi_n(z) \rightarrow 0$ for every $z \in \mathbb{H}$.

We now wish to focus attention on the closed subspace \mathcal{Z}_0 of \mathcal{Z} defined in section 1.6, the tangent vector fields to the symmetric circle maps.

Theorem 9 *The following conditions on an element V of \mathcal{Z} are equivalent:*

1. *with respect to any smooth local coordinate x on the boundary of Ω ,*

$$\left| \frac{V(x+t) - 2V(x) + V(x-t)}{t} \right| \leq c(t),$$

where $c(t)$ approaches 0 as $t \rightarrow 0$,

2. *V has a continuous extension \tilde{V} for which $\bar{\partial} \tilde{V} = \mu$ is vanishing in the sense that for every $\epsilon > 0$ there exists a compact subset of \mathbb{H} such that if z lies outside the compact set then $|\mu(z)| < \epsilon$,*

3. *V annihilates every degenerating sequence in \mathcal{A} ,*

PROOF. Given V satisfying condition 1, the Beurling-Ahlfors formula (19) yields a vector field \tilde{V} with the property that $\bar{\partial} \tilde{V}$ is vanishing. Thus 1 implies 2. It is

easy to see that if $|\mu(z)| < \epsilon$ for z outside a sufficiently large compact set and if φ_n is degenerating, then

$$\lim_{n \rightarrow \infty} \iint \mu \varphi_n dx dy \rightarrow \infty,$$

and so 2 implies 3. To see that 3 implies 1, consider the following sequence of quadratic differentials φ_n , where $t_n \rightarrow 0$ and x_n is arbitrary:

$$\begin{aligned} \varphi_n(z) &= \frac{1}{t_n} \left\{ \frac{1}{z - (x_n - t_n)} - \frac{2}{z - x_n} + \frac{1}{z - (x_n + t_n)} \right\} \\ &= \frac{2t_n}{(z - (x_n - t_n))(z - x_n)(z - (x_n + t_n))}. \end{aligned}$$

Note that

$$\iint_{\mathbb{H}} |\varphi_n| = \iint_{\mathbb{H}} \left| \frac{2t_n}{(z - t_n)z(z + t_n)} \right| dx dy = \iint_{\mathbb{H}} \left| \frac{1}{(z - 1)z(z + 1)} \right| dx dy,$$

which is a positive constant not depending on n and, for fixed z in the upper half-plane, $\varphi_n(z) \rightarrow 0$ as $t_n \rightarrow 0$.

By formula (28)

$$(V, \varphi_n) = \frac{\pi}{2} \left\{ \frac{V(x_n - t_n) - 2V(x_n) + V(x_n + t_n)}{t_n} \right\}$$

and we know that this quantity approaches zero as $t_n \rightarrow 0$, no matter which sequence $\{x_n\}$ is selected. Thus 3 implies 1. \square

Theorem 10 *The pairing (V, φ) defined in (27) induces an isomorphism from \mathcal{A} onto \mathcal{Z}_0^* .*

PROOF. We first observe that the pairing defined in (27) is non-degenerate between \mathcal{Z}_0 and \mathcal{A} . Since it is a non-degenerate pairing between \mathcal{A} and \mathcal{Z} and since $\mathcal{Z}_0 \subset \mathcal{Z}$, whenever $V \in \mathcal{Z}_0$, $(V, \varphi) = 0$ for all $\varphi \in \mathcal{A}$ implies $V = 0$. Moreover, for $|z_0| < 1$ and $\epsilon < 1 - |z_0|$, by the mean value property

$$\varphi(z_0) = \frac{1}{\pi \epsilon^2} \iint_{|z - z_0| < \epsilon} \varphi(z) dx dy = \iint \mu_0 \varphi,$$

where

$$\mu_0(z) = \begin{cases} \frac{1}{\pi \epsilon^2} & \text{for } |z - z_0| \leq \epsilon, \\ 0 & \text{for } z \text{ elsewhere.} \end{cases}$$

Because the pairing is non-degenerate, the mapping from \mathcal{A} to \mathcal{Z}_0^* given by $\varphi \mapsto \{V \mapsto (V, \varphi)\}$ is well-defined and injective. In order to show it is surjective it suffices to show the unit ball of \mathcal{A} is compact with respect to the

weak topology. To this end, assume φ_n is a sequence in \mathcal{A} with $\|\varphi_n\| = 1$ and L is a linear functional of the form

$$L(\varphi) = \iint_{\mathbb{H}} \mu \varphi,$$

where $|\mu(z)| < \epsilon$ for z outside sufficiently large compact subsets of \mathbb{H} . By normal convergence φ_n has a subsequence that converges uniformly on compact subsets to some φ , which (in order to avoid cumbersome notation) we denote also by φ_n . Note that by Lebesgue dominated convergence

$$\lim_{n \rightarrow \infty} \iint (|\varphi_n - \varphi| - |\varphi_n|) = \lim_{n \rightarrow \infty} (|\varphi_n - \varphi| - |\varphi_n|) = -\|\varphi\|,$$

and, since $\|\varphi_n\| = 1$, $\|\varphi_n - \varphi\| \rightarrow 1 - \|\varphi\|$.

We divide the argument into three cases: either $\|\varphi\| = 0$ or $0 < \|\varphi\| < 1$ or $\|\varphi\| = 1$. In the first case, φ_n is degenerating and $L(\varphi_n) \rightarrow 0$. In the second case, since $\|\varphi\| < 1$, if we put

$$\tilde{\varphi}_n = \frac{\varphi_n - \varphi}{\|\varphi_n - \varphi\|},$$

then the denominator is bounded away from zero. Thus $\tilde{\varphi}_n$ is a degenerating sequence and $L(\tilde{\varphi}_n)$ converges to zero, which implies $L(\varphi_n)$ converges to $L(\varphi)$. In the third case $\|\varphi_n - \varphi\| \rightarrow \infty$, and so $L(\varphi_n)$ converges to $L(\varphi)$. Thus, in all cases $L(\varphi_n)$ converges to $L(\varphi)$. \square

DEFINITION. We say a sequence V_n in \mathcal{Z} is *vanishing* if there is a constant $C > 1$ such that $C^{-1} \leq \|V_n\|_T \leq C$ and $V_n(x)$ approaches zero for every x on the boundary.

The following theorem enables one to deduce that an automorphism of \mathcal{Z} that is an isometry for the infinitesimal Teichmüller norm necessarily preserves the closed subspace \mathcal{Z}_0 . This is a key step in the proof of the automorphism theorem, Theorem 6 of section 1.8.

Theorem 11 *An element V of \mathcal{Z} with $\|V\|_T = 1$ is in \mathcal{Z}_0 if, and only if, for every vanishing sequence of elements W_n in \mathcal{Z} with $\|W_n\|_T = 1$,*

$$\limsup_{n \rightarrow \infty} \|V + W_n\|_T \leq \|V\|_T.$$

For the proof we refer to [19] and to [33].

2.4 Reich-Strebel Inequalities

Let μ be the Beltrami coefficient of a quasiconformal extension \tilde{h} of a quasymmetric mapping h , and let $K_0 = K_0(h)$ be the smallest possible dilatation of a

quasiconformal extension of h . For any holomorphic quadratic differential φ in $\mathcal{A}(\mathbb{H})$ with

$$\|\varphi\| = \int \int_{\mathbb{H}} |\varphi(z)| dx dy = 1,$$

one has the following bounds on K_0 :

$$\frac{1}{K_0} \leq \int \int_{\mathbb{H}} \frac{|1 - \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi(z)| dx dy, \quad (29)$$

and

$$K_0 \leq \sup_{\|\varphi\|=1} \int \int_{\mathbb{H}} \frac{|1 + \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi(z)| dx dy. \quad (30)$$

These inequalities were proved by Reich and Strebel [61], [62], who also observed that they yield the infinitesimal form of Teichmüller's metric:

$$d_T([0], [t\mu]) = \frac{1}{2} \log K_0(t\mu) = t \sup_{\|\varphi\|=1} \left| \int \int_{|z|<1} \mu(z) \varphi(z) dx dy \right| + o(t), \quad (31)$$

for $t > 0$. The inequality

$$d_T([0], [t\mu]) \geq t \operatorname{Re} \left\{ \int \int_{|z|<1} \mu \varphi \right\} + o(t),$$

for $\|\varphi\| = 1$, follows by replacing μ by $t\mu$ and calculating the first variation in (29). Similarly, the inequality

$$d_T([0], [t\mu]) \leq t \sup_{\|\varphi\|=1} \operatorname{Re} \left\{ \int \int_{|z|<1} \mu \varphi \right\} + o(t),$$

follows on replacing μ by $t\mu$ and calculating the first variation in (30).

2.5 Tangent Spaces Revisited

Let M denote the open unit ball in $L_\infty(\mathbb{H})$ and suppose μ_t is a smooth curve in M such that each f^{μ_t} is equal to the identity on the boundary of \mathbb{H} . Then from inequality (29) we see that for every holomorphic quadratic differential $\varphi(\zeta) d\zeta^2$ on \mathbb{H} ,

$$1 \leq \int \int_{\mathbb{H}} \frac{|1 - \mu_t \frac{\varphi}{|\varphi|}|^2}{1 - |\mu_t|^2} |\varphi| d\xi d\eta.$$

By putting

$$\|\mu_t - t\nu\|_\infty = o(t) \quad (32)$$

and computing the first variation in this inequality, one obtains

$$\int \int_{\mathbb{H}} \varphi(\zeta) \nu(\zeta) d\xi d\eta = 0, \quad (33)$$

for every such φ .

Conversely, suppose (33) holds for every holomorphic quadratic differential φ on the upper half-plane. Restricting $\varphi(\zeta)d\zeta^2 = \frac{d\zeta^2}{(\zeta-z)^4}$ to the lower half-plane, we conclude that

$$\iint_{\mathbb{H}} \frac{\nu(\zeta)}{(\zeta-z)^4} d\xi d\eta = 0, \quad (34)$$

where $\zeta = \xi + i\eta$ is in the upper half-plane.

A key existence theorem (see [30], page 107) says that (34) implies there exists a curve μ_t such that

$$\|\mu_t - t\nu\|_{\infty} = o(t)$$

and $f^{\mu_t}(z) = z$ for every z in the closure of the lower half-plane. Here, f^{μ_t} is the unique quasiconformal self-map of the whole plane that fixes 0, 1 and ∞ , and that has Beltrami coefficient equal to μ_t in the upper half-plane and identically equal to 0 in the lower half-plane. In particular, if we let M_0 be the closed subspace of those μ in M for which $f^{\mu}(x) = x$ for all $x \in \mathbb{R}$, then the tangent space N to M_0 consists of those ν for which

$$\iint_{\mathbb{H}} \nu(\zeta)\varphi(\zeta)d\xi d\eta = 0$$

for every quadratic differential φ holomorphic in \mathbb{H} . Moreover, the tangent space to Teichmüller space T is isomorphic to the factor space $L_{\infty}(\mathbb{H})/N$, see [30] and [33]

For μ in $L_{\infty}(\mathbb{H})$, define

$$\hat{\mu} = \begin{cases} \mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H} \\ \overline{\mu(\bar{\zeta})} & \text{for } \zeta \text{ in } \mathbb{H}^*. \end{cases}$$

Let $\alpha : L^{\infty}(\mathbb{H}) \rightarrow Z$ be the map $\alpha : \mu \mapsto V_{\mu}(x)$ where x is in \mathbb{R} and

$$V_{\mu}(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\hat{\mu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta. \quad (35)$$

Also define

$$\tilde{\mu}(\zeta) = \begin{cases} \mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H} \\ 0 & \text{for } \zeta \text{ in } \mathbb{H}^*, \end{cases}$$

and let β be the Bers map $\beta : \mu \mapsto (W_{\mu})'''(z)$, where

$$W_{\mu}(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\tilde{\mu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta. \quad (36)$$

Theorem 12 *The maps α and β defined above induce isomorphisms of Banach spaces from $L_\infty(\mathbb{H})/N$ onto \mathcal{Z} and from $L_\infty(\mathbb{H})/N$ onto B , where B is the Banach space of holomorphic functions $\psi(z)$ defined in the lower half-plane \mathbb{H}^* for which*

$$\|\psi\|_B = \sup_{z \in \mathbb{H}^*} |\psi(z)y^2| < \infty.$$

PROOF. Note that

$$\alpha(\mu) = \operatorname{Re} \left(-\frac{2}{\pi} \iint_{\mathbb{H}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\xi d\eta \right),$$

and therefore the condition that $\alpha(\mu)(x) = 0$ for all x in \mathbb{R} implies

$$\iint_{\mathbb{H}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta = 0$$

for all z in the lower half plane. On taking the third derivative with respect to z , we find that

$$\iint_{\mathbb{H}} \frac{\mu(\zeta)}{(\zeta-z)^4} d\xi d\eta = 0,$$

for all z in the lower half-plane. Since finite linear combinations of the form

$$\varphi(\zeta) = \sum_j c_j \frac{1}{(\zeta - z_j)^4},$$

where z_j are points in the lower half-plane, are dense in the space of integrable holomorphic quadratic differentials in the upper half plane (see [2], [6]), we see that $\alpha(\mu)(x) = 0$ for all x in \mathbb{R} implies $\iint_{\mathbb{H}} \varphi(\zeta) \mu(\zeta) d\xi d\eta = 0$ for all φ , which implies μ is in N .

Conversely, if μ is orthogonal to every φ , then

$$\iint_{\mathbb{H}} \frac{\mu(\zeta)}{(\zeta-z)^4} d\xi d\eta = 0,$$

for every z in the lower half plane and, by integrating three times and normalizing so that $V_\mu(z)$ vanishes at 0, 1 and ∞ , we find that

$$V_\mu(x) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\hat{\mu}(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\xi d\eta = 0,$$

for all x in \mathbb{R} .

To see that α is surjective we apply the extension formula (20) to the vector field $V(x) \frac{\partial}{\partial x}$. That is, for given $V(x) \frac{\partial}{\partial x}$ representing an element in \mathcal{Z} , we put $V(z) = W_1(z) + iW_2(z)$, where

$$W_1(x+iy) = \frac{1}{2y} \int_{x-y}^{x+y} V(t) dt, \quad (37)$$

and

$$W_2(x + iy) = \frac{1}{y} \left\{ \int_x^{x+y} V(t) dt - \int_{x-y}^y h(t) dt \right\}. \quad (38)$$

Then it is a routine calculation (see [35]) to show that $\|\frac{\partial}{\partial \bar{z}} V(z)\|_\infty < \infty$.

We leave it to the reader to show that the Bers map β is an isomorphism, and in particular that $\beta(\mu) = 0$ if, and only if, $V_\mu(x) = 0$ for every x in \mathbb{R} . A detailed proof may be found in [33], p.134, or in [48], pp. 111-114. \square

2.6 Hilbert Transform and Almost Complex Structure

For a smooth real-valued function $f(x)$ defined on the real axis with compact support, the Hilbert transform Jf is normally defined as the principal part of a divergent integral. That is,

$$(Jf)(x) = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left\{ \int_{-\infty}^{x-\epsilon} f(t) dt + \int_{x+\epsilon}^{\infty} f(t) dt \right\}. \quad (39)$$

This formula hides a description of the transform in terms of harmonic conjugates which is invariant under conformal changes of coordinate. This description has three steps. The first step, if it is possible, is to form the unique harmonic extension $\tilde{f}(z)$ to the upper half-plane, characterized by the properties that $\tilde{f}(z)$ is harmonic and $\tilde{f}(x)$ coincides with $f(x)$ for x real. Then one forms $\tilde{g}(z)$, which is unique up to an additive constant and such that $\tilde{f}(z) + i\tilde{g}(z)$ is holomorphic in the upper half-plane. Finally, $Jf(x)$ is defined to be the restriction of $\tilde{g}(z)$ to the real axis.

Of course, the definition of Jf in given this way is determined only up to additive constant, but in order for J to be well-defined on \mathcal{Z} , we need only to define JV up to the addition of a quadratic polynomial $p(z) = az^2 + bz + c$.

We wish to give an alternative description of the Hilbert transform on the space \mathcal{Z} . Since α is surjective, we can assume V is of the form

$$V(x) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-x)} d\xi d\eta,$$

where $\mu(z) = \overline{\mu(\bar{z})}$. We shall say that μ satisfying this equation is *symmetric*. For symmetric μ , we define $\hat{\mu}$ to be the Beltrami coefficient given by the formula

$$\hat{\mu}(\zeta) = \begin{cases} i\mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H} \\ -i\mu(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}^*. \end{cases} \quad (40)$$

Then $\hat{\mu}$ is also symmetric and

$$(V_\mu + iV_{\hat{\mu}})(z) = \frac{-2}{\pi} \iint_{\mathbb{H}^*} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta,$$

where the integration is over the lower half-plane \mathbb{H}^* . It is obvious that this function is holomorphic in the upper half-plane; therefore, up to the addition of a quadratic polynomial, JV_μ is the restriction to the real axis of $V_{\hat{\mu}}(z)$.

Note that $\|\hat{\mu}\|_\infty = \|\mu\|_\infty$. This reformulation shows that \mathcal{Z} is invariant under J , and in fact J is an isometry for the infinitesimal Teichmüller norm on the tangent space to Teichmüller space. The argument is easily modified to show that \mathcal{Z}_0 is also invariant under J , see ([35]). It also shows that the Hilbert transform applied to the vector field $V(x)\frac{\partial}{\partial x}$ corresponds to the mapping $\mu \mapsto i\mu$ for Beltrami coefficients given in the upper half-plane. Since multiplication by i on Beltrami coefficients determines the standard almost complex structure on Teichmüller space, the Hilbert transform gives the same almost complex structure. This observation is due to Steven Kerckhoff (unpublished, but see [76]).

2.7 Complex Structures on Quasi-Fuchsian Space

The view of the almost complex structures summarized in the previous section has been exploited by Giannis Platis [60] to yield *three* inter-related but distinct almost complex structures on the quasi-Fuchsian spaces $QF = QF(\Gamma)$. These are complex deformation spaces, whose points are given by arbitrary quasiconformal conjugates of a given (cofinite volume) Fuchsian group Γ , acting discretely on the union of \mathbb{H} and \mathbb{H}^* , the complement of the circle $\overline{\mathbb{R}}$ inside the Riemann sphere. Such a group $H\Gamma H^{-1}$ is known as a *quasi-Fuchsian group*. It operates discretely, but not necessarily symmetrically, on the complement of the quasicircle $H(\overline{\mathbb{R}})$; our earlier definition of the Teichmüller spaces implies that $QF(\Gamma) \supset T(\Gamma)$ as a diagonal subset, corresponding to q-c conjugates where the mapping H is given by a symmetric Beltrami coefficient. Together with a certain hermitian 2-form Ω defined on the space $QF(\Gamma)$, the three anti-involutions determine a hyper-Kählerian structure. One views the tangent space to $QF(\Gamma)$ as a space of (complex) vector fields $V(x)\frac{\partial}{\partial x}$ which can be expressed in the form

$$V(x) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta.$$

In this formula, $V(x)$ is usually complex-valued because there is no assumption about symmetry for μ .

We define I to be the map of vector fields induced by $\mu \mapsto i\mu$. Writing

$$\mu = \begin{cases} \mu_1(\zeta) & \text{for } \zeta \text{ in } \mathbb{H} \text{ and} \\ \mu_2(\zeta) & \text{for } \zeta \text{ in } \mathbb{H}^*, \end{cases}$$

we define $V \mapsto J(V)$ to be the map induced by

$$\mu \mapsto \begin{cases} \overline{\mu_2(\bar{\zeta})} & \text{for } \zeta \text{ in } \mathbb{H} \text{ and} \\ -\overline{\mu_1(\bar{\zeta})} & \text{for } \zeta \text{ in } \mathbb{H}^*. \end{cases}$$

A simple calculation shows that $IJ = -JI$, so that if we write $K = I \circ J$, then I^2 , J^2 and K^2 are all equal to minus the identity and $IJ = K$, $JK = I$ and

$KI = J$. Moreover, K restricted to symmetric Beltrami coefficients coincides with the almost complex structure defined (via the Hilbert transform) in the preceding section on Teichmüller space.

In [60], Platis shows that for finite co-volume Fuchsian groups, I, J , and K together with the hermitian form Ω yield a hyper-Kählerian structure on $QF(\Gamma)$. The form Ω is constructed using derivatives of a finite spanning set of complex length functions (see for instance [44]); it is compatible with the almost complex structure induced on QF as a product space $T(\Gamma) \times \overline{T(\Gamma)}$ by the anti-involution J , satisfies a complex analogue of Wolpert's reciprocity formula for hyperbolic length functions on Teichmüller space, and restricts on the diagonal subspace to give the Weil-Petersson metric on Teichmüller space.

2.8 Automorphisms are Geometric

To close this article we return to the rigidity theorem, Theorem 6, formulated in section 1.8. This result is the analogue for universal Teichmüller space of the classical result of H. Royden [64] and of Earle and Kra [24] that says that any automorphism of the Teichmüller space of a surface of genus greater than 3 and possibly with a finite number of punctures is induced by an element of the mapping class group. That the parallel result holds for any surface of finite genus with a finite number of holes removed was proved in [19] and for any open surface of finite genus by Lakic in [50].

Suppose we are given an almost complex diffeomorphism F of universal Teichmüller space, T . Since Kobayashi's metric coincides with Teichmüller's metric on T [29], the automorphism is an isometry in Teichmüller's metric, and since Teichmüller's metric is the integral of its infinitesimal form [59], this means that if $F([0]) = \tau$, then $F' = dF$ defines an isometry from the tangent space at $[0]$ to the tangent space at τ . Since we may select a geometric isomorphism ρ_h such that $\rho_h \circ F([0]) = [0]$ and since geometric isomorphisms are isometries, we obtain an automorphism $\rho_h \circ F$ which preserves the basepoint $[0]$ and induces an isometry on the tangent space \mathcal{Z} at $[0]$ to Teichmüller space. One shows that this isometry is necessarily equal to the identity and thus $F = \rho_{h^{-1}}$, that is, every automorphism of Teichmüller space is induced by the action of a quasisymmetric mapping on the boundary of the hyperbolic plane.

We outline the key steps in the proof. One first shows that any isometry I of \mathcal{Z} with Teichmüller's infinitesimal metric must be induced by an isometry of the predual space A . This result follows from the results of section 2.3, and in particular from Theorem 11. Such an isometry must preserve the closed subspace \mathcal{Z}_0 and therefore it is equal to the second dual of its restriction to \mathcal{Z}_0 . Thus there is an isometry \hat{I} of \mathcal{A} such that I is the dual of \hat{I} under the natural pairing between A and \mathcal{Z} . Then one shows that \hat{I} is induced by the composition of multiplication by a complex constant c (of modulus 1 since it is an isometry) with a conformal map. That is,

$$\hat{I}(\varphi) = c\varphi(f(z))f'(z)^2,$$

where f is a conformal self-map of the base Riemann surface. For universal

Teichmüller space, the base Riemann surface is the upper half-plane and so f is a real Möbius transformation in this case. Finally, one shows that the constant c is equal to 1 : for this step, see [19] or [33].

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